9. É. Shternberg, Load Transmission and Load Diffusion in Statics of an Elastic Body, Sb. Per. Mekhanika, No. 6 (1972).
10. L. I. Manevich and A. V. Pavlenko, "Three-dimensional problem in the theory of elasticity for anisotropic media," in: All-Union Conf. on the Theory of Elasticity, Abstracts of Reports, Erevan (1979).
11. N. I. Vorob'eva, S. G. Koblik, and L. I. Manevich, "Axisymmetrical contact problem taking into account adhesion and slipping," Prikl. Mat. Mekh., 43, No. 3 (1979).

STABILITY OF PLASTIC ELONGATION OF A BIMETALLIC SHEET

S. S. Oding

UDC 539.214:539.374

The extension of a sheet is limited by the magnitude of the critical strain at which local thinning of the material begins with the formation of a neck. We solve the problem of the stability of plastic extension of a bimetallic sheet under conditions of plane strain. The solution is constructed by using the theory of finite deformations of a rigid-plastic material.

1. We consider the plastic extension of a bimetallic sheet with a given law of variation of length. The loss of stability in this case can be represented as a process of continuous change of equilibrium shapes. Therefore the critical strain at which a neck is produced can be determined by the bifurcation method.

Since the loss of stability of deformation under consideration occurs during the plastic deformations developed, we neglect elastic deformations and assume a model of a rigid-plastic materials with isotropic hardening.

The flow curves of the materials of the layers of the bimetallic sheet $\sigma_e^{(1)} = \sigma_e^{(1)}(e_e)$ and $\sigma_e^{(2)} = \sigma_e^{(2)}(e_e)$ are assumed given. Here superscripts 1 and 2 denote quantities referring to the separate layers of the sheet; $\sigma_e = [(3/2)s_{ij}s_{ij}]^{1/2}$, stress intensity; $s_{ij} = \sigma_{ij} - \sigma\delta_{ij}$, components of the stress deviator; σ_{ij}, components of the stress tensor; $\sigma = (1/3)\sigma_{mn}\delta_{mn}$, hydrostatic pressure; and e_e, cumulative plastic deformation.

The problem is to determine the strain e_{e*} beyond which deformation occurs with the formation of a neck.

As the equations of state we take the equations of the deformation theory of plasticity, written for finite strains in the form

$$s_{ij} = \frac{2}{3}\frac{\sigma_e}{e_e}e_{ij}, \tag{1.1}$$

where the e_{ij} are the logarithmic strains; $e_e = [(2/3)e_{ij}e_{ij}]^{1/2}$ is the intensity of the logarithmic strains. Logarithmic strains are used in Eqs. (1.1), since for large deformations the condition of incompressibility of the material $e_{ij}\delta_{ij} = 0$ is compatible with Eqs. (1.1) only for logarithmic strains.

Bifurcation in a state A means that in addition to unperturbed deformation in state A, perturbed deformation in a state B infinitely close to it is possible.

We introduce a Cartesian coordinate system in state A in such a way that for the same particles the coordinates x_i' in state B and the coordinate x_i in state A are related by the equation

$$x_i' = x_i + u_i, \tag{1.2}$$

where u_i is an additional infinitesimal displacement of a particle. Then

$$dx_i' = (\delta_{ij} + u_{i,j})\,dx_j. \tag{1.3}$$

From now on subscripts after a comma denote differentiation with respect to the corresponding coordinate x_i.

Retaining only first order infinitesimals in the determinant of system (1.3), we obtain the incompressibility condition of the medium in the form

Voronezh. Translated from Zhurnal Prikladnoi Mekhaniki i Tekhnicheskoi Fiziki, No. 3, pp. 146-150, May-June, 1982. Original article submitted March 23, 1981.

$$u_{k,k}=0. \tag{1.4}$$

The derivative of some quantity a with respect to x_i' is expressed in terms of its derivative with respect to x_i in the form

$$\partial a/\partial x_i'=(\delta_{ik}-u_{k,i})\,a_{,k}. \tag{1.5}$$

Comparing states A and B, we write all quantities in the perturbed state B in the form

$$\sigma_{ij}'=\sigma_{ij}^0+\sigma_{ij},\ e_{ij}'=e_{ij}^0+e_{ij},\ u_i'=u_i^0+u_i,$$

where σ_{ij}, e_{ij}, and u_i, are, respectively, additional infinitesimal components of the stress tensor, the logarithmic strain tensor, and the displacement. A subscript 0 denotes a quantity referring to the unperturbed state A. In addition to the logarithmic strain tensor e_{ij}', we consider the Almansi tensor of finite deformations coaxial with it,

$$\varepsilon_{ij}'=\frac{1}{2}\left(\frac{\partial u_i'}{\partial x_j'}+\frac{\partial u_j'}{\partial x_i'}-\frac{\partial u_k'}{\partial x_i'}\frac{\partial u_k'}{\partial x_j'}\right). \tag{1.6}$$

As a result of linearization of (1.6), we obtain by using (1.5)

$$\varepsilon_{ij}=\frac{1}{2}(u_{i,j}+u_{j,i})-(u_{i,k}^0u_{k,j}+u_{j,k}^0u_{k,i})\ +(u_{n,i}^0u_{n,m}^0u_{m,j}+u_{n,j}^0u_{n,m}^0u_{m,i}). \tag{1.7}$$

The relation between the principal components and the principal additional components of these strain tensors has the form

$$e_k'=-\frac{1}{2}\ln(1-2\varepsilon_k');\ e_k=\varepsilon_k/(1-2\varepsilon_k^0). \tag{1.8}$$

From now on by restricting ourselves to a homogeneous subcritical state and superposing the x_i axes on the principal directions of the strain tensor of the unperturbed state, we have

$$\begin{aligned} e_{ij}&=\varepsilon_{ij}/(1-2\varepsilon_i^0)\ \text{for } i=j\ (\text{no } \Sigma \text{ over } i,\ j),\\ e_{ij}&=\varepsilon_{ij}(e_i^0-e_j^0)/(\varepsilon_i^0-\varepsilon_j^0)\quad\text{for}\quad i\neq j. \end{aligned} \tag{1.9}$$

In addition, we rewrite (1.7) in the form

$$\varepsilon_{ij}=\frac{1}{2}\left[(1-2\varepsilon_i^0)\,u_{i,j}+(1-2\varepsilon_j^0)\,u_{j,i}\right]\ (\text{no} \sum \text{over } i,\ j). \tag{1.10}$$

Substituting (1.10) into (1.9), we have

$$e_{ij}=u_{i,j}\quad\text{for}\quad i=j,\ e_{ij}=B_{ij}u_{i,j}+B_{ji}u_{j,i}\quad\text{for}\quad i\neq j\ (\text{no } \Sigma \text{ over } i,\ j), \tag{1.11}$$

where

$$B_{ij}=\frac{1}{2}(1-2\varepsilon_i^0)\frac{e_i^0-e_j^0}{\varepsilon_i^0-\varepsilon_j^0}=\frac{\lambda_j^2}{\lambda_i^2-\lambda_j^2}\ln\frac{\lambda_i}{\lambda_j},\ e_i^0=\ln\lambda_i.$$

Writing the equations of equilibrium of an element of the medium in states A and B, subtracting one from the other, and using (1.5), we obtain

$$\sigma_{ij,j}-\sigma_{ij,k}^0u_{k,j}=0. \tag{1.12}$$

As a result of linearizing the equations of state (1.1), we have in our notation

$$s_{ij}=\frac{1}{K}\left\{\frac{2}{3}e_{ij}+A_{ij}(\eta K-1)\,e_e\right\}, \tag{1.13}$$

where

$$\sigma_e=\eta e_e;\ K=\frac{e_e^0}{\sigma_e^0};\ A_{ij}=\frac{s_{ij}^0}{\sigma_e^0}=\frac{2}{3}\frac{e_{ij}^0}{e_e^0}.$$

The additional strain intensity is

$$e_e=A_{mn}e_{mn}. \tag{1.14}$$

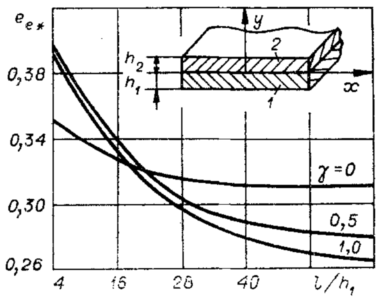

Fig. 1

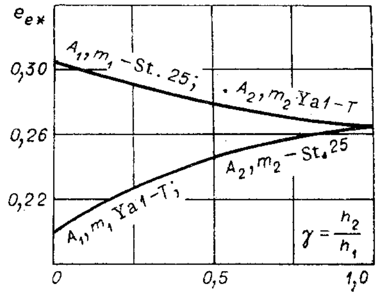

Fig. 2

We rewrite the boundary conditions from [1] for an incompressible material in our notation

$$\left(\sigma_{ij}-\sigma^0_{ki}u_{j,k}\right)\nu^0_j dS=\delta\left(P_i dS\right), \tag{1.15}$$

where dS is an element of area of the surface; ν^0_j, components of a unit vector normal to the surface of the body in the unperturbed state; $P_i dS$, components of the surface load vector.

2. We investigate the stability of the biaxial extension of a bimetallic sheet under conditions of plane strain. The problem of the plane extension of a uniform sheet was solved in [2] in the approximate Leibenzon–Ishlinskii formulation.

Let us consider the solution of the problem for a particular layer of the sheet. The stresses σ^0_x and σ^0_z develop in the x and z directions in an extended layer of the sheet. The y axis is directed along the normal to the surface of the sheet. We assume that the subcritical state is plane and homogeneous:

$$\sigma^0_y=s^0_{xy}=s^0_{xz}=s^0_{yz}=0,\ \sigma^0_z=\sigma^0_x/2;\ e^0_z=0,\ e^0_e=\frac{2}{\sqrt{3}}e^0_x.$$

As in [2], we assume that the sheet loses stability by the formation of a rather extended neck along the z axis. Under these assumptions Eqs. (1.4) and (1.11)-(1.14) reduce to the single equation

$$\left(\frac{\partial^4}{\partial y^4}+2\alpha\frac{\partial^4}{\partial x^2\partial y^2}+\beta^2\frac{\partial^4}{\partial x^4}\right)u_y=0, \tag{2.1}$$

where

$$\alpha=\frac{\eta K}{B_{xy}}-\frac{1}{2}\left(1-\lambda_x^4\right);\ \beta=\lambda_x^4,\ e^0_x=\ln\lambda_x.$$

We seek the solution of this equation in the form

$$u_y=\varphi(t)\cos\lambda x,\ t=\lambda y. \tag{2.2}$$

It follows from the incompressibility condition (1.4) that

$$u_x=-\varphi'(t)\sin\lambda x.$$

Denoting by h_1 and h_2 the thicknesses of the layers of the bimetallic sheet at the instant in question, we transform to dimensionless coordinates. We place the origin of coordinates at the boundary between the layers. Then for the first (lower) layer of the sheet $-1\le y\le 0$, and for the second (upper) layer $0\le y\le\gamma$, where $\gamma=h_2/h_1$.

Under condition (2.2) the solution of Eqs. (2.1) for the first and second layers of the sheet has the form

$$\varphi_1=\left(C_{11}\cos\nu_1 t+C_{12}\sin\nu_1 t\right)e^{\mu_1 t}+\left(C_{13}\cos\nu_1 t+C_{14}\sin\nu_1 t\right)e^{-\mu_1 t},$$
$$\varphi_2=\left(C_{21}\cos\nu_2 t+C_{22}\sin\nu_2 t\right)e^{\mu_2 t}+\left(C_{23}\cos\nu_2 t+C_{24}\sin\nu_2 t\right)e^{-\mu_2 t},$$

where

$$\mu_1=\sqrt{(\beta+\alpha_1)/2};\ \nu_1=\sqrt{(\beta-\alpha_1)/2};$$
$$\mu_2=\sqrt{(\beta+\alpha_2)/2};\ \nu_2=\sqrt{(\beta-\alpha_2)/2};$$

$$\alpha_1 = \frac{\eta_1 K_1}{B_{xy}} - \frac{1}{2}(1-\lambda_x^4);\ \alpha_2 = \frac{\eta_2 K_2}{B_{xy}} - \frac{1}{2}(1-\lambda_x^4).$$

We determine the integration constants C_{ij}, from the condition that there are no loads on the free surfaces of the sheets at $y = -1$ and $y = \gamma$, and from the condition of continuity of the components of the displacements u_x', u_y' and the components of the stresses σ_y', s_{xy}' at the boundary between the two layers (at $y = 0$).

From (1.15) we obtain at $y = -1$

$$s_{xy}^{(1)} - \frac{2}{3}\sigma_e^{0(1)} \frac{\partial}{\partial x} u_y^{(1)} = 0,\ \sigma_y^{(1)} = 0;$$

at $y = \gamma$

$$s_{xy}^{(2)} - \frac{2}{3}\sigma_e^{0(2)} \frac{\partial}{\partial x} u_y^{(2)} = 0,\ \sigma_y^{(2)} = 0;$$

at $y = 0$

$$s_{xy}^{(1)} - \frac{2}{3}\sigma_e^{0(1)} \frac{\partial}{\partial x} u_y^{(1)} = s_{xy}^{(2)} - \frac{2}{3}\sigma_e^{0(2)} \frac{\partial}{\partial x} u_y^{(2)},\ \sigma_y^{(1)} = \sigma_y^{(2)}.$$

These conditions form a system of eight homogeneous algebraic equations for the eight constants C_{ij}. In order for a nontrivial solution to exist, the determinant of the system must vanish.

Since the law of variation of the length of the sheet is given, we determine the parameter λ characterizing the extent of the neck from the condition that the axial displacement u_x at the ends of the sheet is unperturbed, i.e., $u_x = 0$ at $x = \pm l/2$. Here l is the instantaneous length of the sheet.

We use a power law approximation of the flow curves of the materials of the layers of the sheet

$$\sigma_e^{(1)} = A_1 e_e^{m_1},\ \sigma_e^{(2)} = A_2 e_e^{m_2}.$$

A calculation was performed for a bimetallic sheet of Ya 1-T steel ($A = 1010\ \mathrm{MN/m^2}$, $m = 0.263$) and St. 25 steel ($A = 690\ \mathrm{MN/m^2}$, $m = 0.170$). Figure 1 shows the critical strain e_{e*} as a function of the dimensionless length l/h_1 of the bimetallic sheet for several ratios of the thicknesses of the layers. It can be seen from the figure that while for the extension of a continuous sheet ($\gamma = 0$) the critical strain varies with the length only for short lengths, for the extension of bimetallic sheets the length has a significant effect on the critical strain e_{e*} at the instant a neck is formed. Figure 2 shows the effect of the ratio of the thicknesses γ of the layers of the bimetallic sheet on the critical strain for infinitely long sheets ($l \to \infty$ or $\lambda \to 0$). Analysis of the results showed that for infinitely long bimetallic sheets, as for continuous sheets, the critical strain obtained agreed with the strain determined from the condition of reaching the maximum of the axial load.

LITERATURE CITED

1. L. I. Balabukh and M. G. Yakovenko, "On taking account of deformational anisotropy in problems of the stability of isotropic elastic bodies," in: The Mechanics of Deformable Bodies and Structures [in Russian], Mashinostroenie, Moscow (1975).
2. G. D. Del' and S. S. Oding, "The stability of biaxial plastic extension of an isotropic sheet," Izv. Akad. Nauk SSSR, Mekh. Tverd. Tela, No. 3 (1976).